One-dimensional equations with the maximum number of symmetry generators

1987 J. Phys. A: Math. Gen. 20 L701

(http://iopscience.iop.org/0305-4470/20/11/005)

J. Phys. A: Math. Gen. 20 (1987) L701-L704. Printed in the UK

LETTER TO THE EDITOR

One-dimensional equations with the maximum number of symmetry generators

L G S Duarte, S E S Duarte and I C Moreira

Departamento de Física Teórica, Instituto de Física, Universidade Federal do Rio de Janeiro, Ilha do Fundão-21941, Rio de Janeiro, Brazil

Received 17 March 1987

Abstract. In this letter we employ a general space and time transformation to find a class of second-order differential equations with the maximum number (eight) of Lie symmetry generators. We apply this transformation to obtain the symmetry generators for the harmonic oscillator and for a non-linear equation, introduced by Leach, starting from the symmetry generators of the free-particle equation.

Recently there has been increasing interest in studying the symmetries of differential equations modelling physical systems. The method introduced by Lie (1891) considers the invariance of the form of the differential equation itself under point transformations of one parameter. It has been applied, in recent years, to several equations of motion for dynamical systems: the harmonic oscillator (Wulfman and Wybourne 1976), the time-dependent oscillator (Prince and Eliezer 1980), the Kepler problem (Prince and Eliezer 1981), the particle in a constant magnetic field (Moreira 1983), the charge-monopole interaction (Moreira *et al* 1985), etc.

Lie himself showed that for the one-dimensional free particle there are eight point transformations of one parameter that maintain the invariance of the equation; the same situation occurs for a time-dependent oscillator. This is the maximum number of generators for a second-order differential equation of the form

$$\ddot{x}+f(\dot{x}, x, t)=0. \tag{1}$$

In a recent paper Leach (1985) has obtained an 'unexpected result': the non-linear equation

$$\ddot{x}+2x\dot{x}+\tfrac{4}{9}x^3=0 \tag{2}$$

also has eight symmetry generators. In this letter we obtain a general class of one-dimensional equations with the same property: all of them have the maximum number of symmetry generators for a second-order differential equation. We generate this class starting from the idea that this kind of equation can be transformed, by a point transformation, to the free-particle equation. We also show that this transformation permits us to obtain directly the symmetry generators for this class of equations by using the symmetry generators of the free particle. In particular, we apply this procedure for the harmonic oscillator and equation (2).

We start from the free-particle equation

$$\mathrm{d}^2X/\mathrm{d}T^2=0. \tag{3}$$

If we make an invertible point transformation

$$\begin{aligned} X &= F(x, t) \qquad & x &= P(X, T) \\ T &= G(x, t) \qquad & t &= Q(X, T) \end{aligned} \tag{4}$$

with $\Delta \equiv G_t F_x - G_x F_t \neq 0$, then equation (3) will have the form

$$\ddot{x} + \Lambda_3(x, t)\dot{x}^3 + \Lambda_2(x, t)\dot{x}^2 + \Lambda_1(x, t)\dot{x} + \Lambda_0(x, t) = 0 \tag{5}$$

where

$$\begin{aligned} \Lambda_3 &= (G_x F_{xx} - F_x G_{xx})/\Delta \\ \Lambda_2 &= (G_t F_{xx} + 2G_x F_{tx} - 2F_x G_{tx} - F_t G_{xx})/\Delta \\ \Lambda_1 &= (G_x F_{tt} + 2G_t F_{tx} - 2F_t G_{tx} - F_x G_{tt})/\Delta \\ \Lambda_0 &= (G_t F_{tt} - F_t G_{tt})/\Delta. \end{aligned} \tag{6}$$

The eight symmetry generators for equation (3) are

$$\begin{aligned} &U_1 = \partial/\partial T \qquad U_2 = \partial/\partial X \qquad U_3 = T\,\partial/\partial T \qquad U_4 = X\,\partial/\partial X \\ &U_5 = X\,\partial/\partial T \qquad U_6 = T\,\partial/\partial X \qquad U_7 = T^2\,\partial/\partial T + TX\,\partial/\partial X \\ &U_8 = X^2\,\partial/\partial X + TX\,\partial/\partial T. \end{aligned} \tag{7}$$

By using the transformation (4) we obtain, from equation (7), the symmetry generators for equation (5):

$$\begin{aligned} &U_1 = Q_T(x, t)\partial/\partial t + P_T(x, t)\partial/\partial x \qquad && U_2 = Q_X\partial/\partial t + P_X\partial/\partial x \\ &U_3 = GQ_T\partial/\partial t + GP_T\partial/\partial x && U_4 = FQ_X\partial/\partial t + FP_X\partial/\partial x \\ &U_5 = FQ_T\partial/\partial t + FP_T\partial/\partial x && U_6 = GQ_X\partial/\partial t + GP_X\partial/\partial x \\ &U_7 = (G^2 Q_T + GFQ_X)\partial/\partial t + (G^2 P_T + GFP_X)\partial/\partial x \\ &U_8 = (GFP_T + F^2 P_X)\partial/\partial x + (GFQ_T + F^2 Q_X)\partial/\partial t. \end{aligned} \tag{8}$$

We now analyse particular cases of equation (5). If we let $\Lambda_3 = 0$, equation (6) yields

$$G = f_1(t)F(x, t) + g_1(t). \tag{9}$$

Also letting $\Lambda_2 = 0$, we find that two types of solution for F and G are

$$\text{(i)}\quad F = a(t)x + b(t) \qquad G = C(ax + b) + g_1(t) \tag{10}$$

$$\text{(ii)}\quad F = \frac{1}{p(t)x + q(t)} - \frac{\dot{g}_1}{\dot{f}_1} \qquad G = \frac{f_1}{px + q} - \frac{\dot{g}_1 f_1}{\dot{f}_1} + g_1 \tag{11}$$

where C is a constant.

By using equation (6) we obtain, for these two solutions,

$$\text{(i)}\quad \Lambda_1 = \frac{2\dot{a}\dot{g}_1 - a\ddot{g}_1}{a\dot{g}_1} \qquad \Lambda_0 = \frac{x(\ddot{a}\dot{g}_1 - \dot{a}\ddot{g}_1) + (\ddot{b}\dot{g}_1 - \dot{b}\ddot{g}_1)}{a\dot{g}_1} \tag{12}$$

$$\text{(ii)}\quad \Lambda_1 = 2\dot{f}_1(\dot{p}x + 2p\dot{q} - \dot{p}q) + \ddot{f}_1 + h\dot{f}_1(px + q)$$

$$\Lambda_0 = \frac{h^2}{p}(px + q)^3 - \frac{(\dot{h}_1\dot{f}_1 - h\ddot{f}_1)}{p\dot{f}_1}(px + q)^2 - \frac{3h}{p}(\dot{p}x + \dot{q})(px + q) - \frac{\ddot{f}_1}{p\dot{f}_1}(\dot{p}x + \dot{q}) + \frac{(\ddot{p}x + \ddot{q})}{p} \tag{13}$$

where $h=(\dot{g}_1\ddot{f}_1-\ddot{g}_1\dot{f}_1)/(\dot{f}_1)^2$.

Consider first the case of the time-dependent oscillator

$$\ddot{x}+\omega^2(t)x=0. \tag{14}$$

From equations (5) and (12) we find, with $a(t)=\rho^{-1}(t)$,

$$\begin{aligned} F&=\rho^{-1}(t)x+C_1\int\rho^{-2}\,\mathrm{d}t+C_2 \\ G&=C\rho^{-1}(t)x+C_3\int\rho^{-2}\,\mathrm{d}t+C_4 \end{aligned} \tag{15}$$

where $\rho(t)$ satisfies

$$\ddot{\rho}+\omega^2(t)\rho=0. \tag{16}$$

For the time-independent harmonic oscillator, $\omega^2(t)=\omega_0^2$, equation (16) has the solution $\rho=\mathrm{e}^{-\mathrm{i}\omega_0 t}$ and the transformations (15) become

$$\begin{aligned} F&=x\,\mathrm{e}^{\mathrm{i}\omega_0 t}+C_1\,\mathrm{e}^{2\mathrm{i}\omega_0 t}/2\mathrm{i}\omega_0+C_2 \\ G&=Cx\,\mathrm{e}^{\mathrm{i}\omega_0 t}+C_3\,\mathrm{e}^{2\mathrm{i}\omega_0 t}/2\mathrm{i}\omega_0+C_4. \end{aligned} \tag{17}$$

The inverse transformations will be

$$P=X(2\mathrm{i}\omega_0 t)^{-1/2} \qquad Q=\ln(2\mathrm{i}\omega_0 T)/2\mathrm{i}\omega_0 \tag{18}$$

(with $C_1=C_2=C=C_4=0$, $C_3=1$).

We obtain directly from equations (8), (17) and (18) the symmetry generators for the harmonic oscillator (see, for example, Wulfman and Wybourne (1976)).

In the form $\rho=A\cos\omega t$, equation (15) becomes

$$\begin{aligned} F&=x\sec(\omega_0 t)+C_1\tan(\omega_0 t)/\omega_0+C_2 \\ G&=Cx\sec(\omega_0 t)+C_3\tan(\omega_0 t)/\omega_0+C_4. \end{aligned} \tag{19}$$

This transformation, when $C_1=C_2=C_4=0$ and $C_3=1$, is called the Jackiw transformation (Jackiw 1980). It reduces the harmonic oscillator equation to the free-particle equation. The generalisation of this kind of transformation for the case of a time-dependent oscillator with a linear friction $\Lambda_1(t)\dot{x}$ is straightforward. All the infinitesimal symmetry groups for the linear systems considered by Aguirre and Krause (1984) can be found by this method. For example, the symmetry generators for the free-falling particle, $\ddot{x}+g=0$, can be found directly from equation (8) by using the transformation

$$F=x+\tfrac{1}{2}gt^2 \qquad G=t.$$

These point transformations can also be used to obtain the quantum propagator for the time-dependent oscillator (or for more general systems) starting from the free-particle propagator (Junker and Inomata 1985).

From (13) we can show easily that the equation

$$\ddot{x}+Kx\dot{x}+\tfrac{1}{9}K^2x^3=0 \tag{20}$$

is transformed in the free-particle equation if we let

$$F=t/x-\tfrac{1}{6}Kt^2 \qquad G=1/x-\tfrac{1}{3}Kt. \tag{21}$$

Putting $K=2$ we obtain equation (2) introduced by Leach (1985). (The referee has observed that equation (20) can be generalised under the transformation $x \to \alpha X$, $t \to \beta T$.)

The following symmetry generators for (20) are obtained from (8):

$$X_1 = -xt\,\partial/\partial t + (-x^2 + \tfrac{1}{3}Kx^3 t)\,\partial/\partial x \qquad X_2 = x\,\partial/\partial t - \tfrac{1}{3}Kx^3\,\partial/\partial x$$

$$X_3 = t(\tfrac{1}{3}Kxt - 1)\,\partial/\partial t - x(1 + \tfrac{1}{9}K^2x^2t^2 - \tfrac{2}{3}Kxt)\,\partial/\partial x$$

$$X_4 = t(1 - \tfrac{1}{6}Kxt)\,\partial/\partial t + Kx^2t(\tfrac{1}{18}Kxt - \tfrac{1}{3})\,\partial/\partial x$$

$$X_5 = t^2(\tfrac{1}{6}Kxt - 1)\,\partial/\partial t + xt(\tfrac{1}{2}Kxt - 1 - \tfrac{1}{18}K^2x^2t^2)\,\partial/\partial x$$

$$X_6 = (1 - \tfrac{1}{3}Kxt)\,\partial/\partial t + \tfrac{1}{3}Kx^2(\tfrac{1}{3}Kxt - 1)\,\partial/\partial x$$

$$X_7 = \tfrac{1}{6}Kt^2(1 - \tfrac{1}{3}Kxt)\,\partial/\partial t + (\tfrac{2}{3}Kxt - 1 - \tfrac{1}{6}K^2x^2t^2 + \tfrac{1}{54}K^3x^3t^3)\,\partial/\partial x$$

$$X_8 = \tfrac{1}{6}Kt^3(1 - \tfrac{1}{6}Kxt)\,\partial/\partial t + t(\tfrac{1}{2}Kxt - 1 - \tfrac{1}{9}K^2x^2t^2 + \tfrac{1}{108}K^3x^3t^3)\,\partial/\partial x.$$

We also note that the solution $X = AT + B$ for equation (3) permits us to find solutions for the transformed equation (5). For example, by using equation (21) and its inverted transformation, we obtain the following solution for equation (20):

$$x = (t - A)/(\tfrac{1}{6}Kt^2 - \tfrac{1}{3}AKt + 3)$$

where A and B are constants.

This same solution was found by Leach (1985), for the case $K=2$, by direct integration.

The transformation technique employed here to find a class of one-dimensional equations with the maximum number of symmetry generators can be generalised to multidimensional equations. This procedure can be useful to obtain directly the Lie symmetry group of the equations and to construct the quantum propagator starting from the results for the free particle.

We express our appreciation to L P Bueno, O M Ritter, F C Santos and A Tort for stimulating discussions and to P G L Leach for supplying a preprint of his 1985 paper. We would like to thank the referee for some useful suggestions.

References

Aguirre M and Krause J 1984 *J. Math. Phys.* **25** 210
Jackiw R 1980 *Ann. Phys., NY* **129** 183
Junker G and Inomata A 1985 *Phys. Lett.* **110A** 195
Leach P G L 1985 *J. Math. Phys.* **26** 2510
Lie S 1891 *Vorlesunger uber Differentialgleichunger* (Leipzig: Teubner)
Moreira I C 1983 *Suppl. Cie. Cult.* **35** 252
Moreira I C, Ritter O M and Santos F C 1985 *J. Phys. A: Math. Gen.* **18** L427
Prince G E and Eliezer C J 1980 *J. Phys. A: Math. Gen.* **13** 815
—— 1981 *J. Phys. A: Math. Gen.* **14** 587
Wulfman C E and Wybourne B G 1976 *J. Phys. A: Math. Gen.* **9** 507